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# COMPARISON OF BUCKLING DEFORMATIONS IN COMPRESSED RIGID–PLASTIC COSSERAT PLATES WITH THREE-DIMENSIONAL PLATES

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**Abstract**—A comparison is made of solutions for buckling in bending and bulging modes in compressed rigid–plastic Cosserat plates and three-dimensional plates. Close agreement is found in the bending mode, and a value for the constitutive coefficient for transverse shear deformation in plastic bending of Cosserat plates is determined. Comparison of solutions for the bulging mode discloses an anomaly. When a Cosserat plate is loaded in tension instead of compression, the solution for the bulging mode could be expected to describe necking instability However, no solution was found for necking.

## 1. INTRODUCTION

The inception of necking and buckling deformations in rectangular elastic–plastic plates treated as three-dimensional bodies has received considerable attention. Papers by Hill and Hutchinson[1], Miles[2] and Young[3] provide rather comprehensive accounts of various aspects of the problem and include references to earlier work. The effect of transverse shear deformation on the critical compressive load in the buckling of rectangular elastic–plates has been considered in the context of a plate theory by Shrivastava[4]. The present investigation of plastic buckling deformations differs from those just cited, in that the growth of a perturbation is analyzed, rather than bifurcation from a homogeneous state. The present approach has its origin in the work of Goodier and collaborators. A solution for dynamic plastic buckling of thin rectangular plates[5] illustrates the method of representing buckling deformations as small perturbations superposed on continuing uniform flow. In perturbation analysis of plastic buckling, elastic strains are not essential and can be neglected.

In the following, the general theory of an elastic–plastic Cosserat surface due to Green, Naghdi and Osborn[6] is specialized to treat buckling deformations in rectangular rigid–plastic Cosserat plates under uniform uniaxial compression by the method of superposing small deformations on continuing uniform flow. The corresponding deformations in three-dimensional plates are considered next using a general theory of an elastic–plastic continuum, based on the work of Green and Naghdi[7]. Comparison of solutions for sinusoidal bending modes in compressed Cosserat plates and three-dimensional plates shows close agreement, and a value for the constitutive coefficient for transverse shear deformation in plastic bending of Cosserat plates is determined.

Both rigid–plastic Cosserat plates and three-dimensional plates exhibit instability in a symmetric bulging mode. However, in this mode, there is not close agreement in the solutions. In certain details, the response of Cosserat plates is anomalous compared to three-dimensional plates. The various results obtained in the present investigation closely parallel those presented concurrently in a comparison of buckling deformations in elastic Cosserat plates and three-dimensional plates[8].

## 2. KINEMATIC VARIABLES, CONSTITUTIVE RELATIONS AND EQUILIBRIUM CONDITIONS FOR A RIGID–PLASTIC COSSERAT SURFACE

The essential results from the general theory of an elastic–plastic Cosserat surface[6] are now summarized. The set of thirteen kinematic variables,

$$e_{\alpha\beta}, \qquad \kappa_{\alpha\beta}, \qquad \kappa_{3\beta}, \qquad \delta_{\alpha}, \qquad \delta_{3}, \tag{2.1}$$

where the Greek indices take the values 1, 2, are the strains. In an elastic–plastic plate or shell, plastic strains are introduced in addition to the strains (2.1). Here the idealization of rigid–plastic material response is introduced at the outset, so no distinction between the two sets of strains need be made. The strains (2.1) can be expressed in terms of the first and second fundamental forms of the surface, and the director **d**. Referred to a system of convected coordinates $\theta^\alpha$ on the surface, the current values of the coefficients in the first and second fundamental forms, and the surface components of **d**, are denoted, respectively, by

$$a_{\alpha\beta}, \qquad b_{\alpha\beta}, \qquad d_\alpha, \tag{2.2}$$

while their values in a reference state are

$$A_{\alpha\beta}, \qquad B_{\alpha\beta}, \qquad D_\alpha. \tag{2.3}$$

The director **d** also has a normal component $d_3$, which has the value $D_3$ in the reference state. It is convenient to introduce auxiliary kinematic variables $\lambda_{\alpha\beta}$, $\lambda_{3\beta}$ by

$$\lambda_{\alpha\beta} = d_{\alpha|\beta} - b_{\alpha\beta} d_3, \qquad \lambda_{3\beta} = d_{3,\beta} + b^\alpha_\beta d_\alpha. \tag{2.4}$$

In eqn (2.4), the vertical stroke | denotes covariant differentiation with respect to $a_{\alpha\beta}$, the comma denotes partial differentiation, and indices are raised using $a^{\alpha\beta}$, the conjugate of $a_{\alpha\beta}$. In the reference state, $\lambda_{\alpha\beta}$, $\lambda_{3\beta}$ have the values $\Lambda_{\alpha\beta}$, $\Lambda_{3\beta}$. Then the strains (2.1) are given by

$$\begin{aligned} 2e_{\alpha\beta} &= a_{\alpha\beta} - A_{\alpha\beta}, \\ \kappa_{\alpha\beta} &= \lambda_{\alpha\beta} - \Lambda_{\alpha\beta}, \qquad \kappa_{3\beta} = \lambda_{3\beta} - \Lambda_{3\beta}, \\ \delta_\alpha &= d_\alpha - D_\alpha, \qquad \delta_3 = d_3 - D_3. \end{aligned} \tag{2.5}$$

Corresponding to the strains (2.1) are force and couple resultants

$$\overline{N}^{\alpha\beta}, \qquad M^{\alpha\beta}, \qquad M^{3\beta}, \qquad m^\alpha, \qquad m^3, \tag{2.6}$$

such that the mechanical power $P$ is given by

$$P = \overline{N}^{\alpha\beta}\dot{e}_{\alpha\beta} + M^{\alpha\beta}\dot{\kappa}_{\alpha\beta} + M^{3\beta}\dot{\kappa}_{3\beta} + m^\alpha\dot{\delta}_\alpha + m^3\dot{\delta}_3, \tag{2.7}$$

where $P$ is the mechanical power per unit area of surface in the current deformed configuration, and the superposed dot denotes differentiation with respect to time $t$, the material coordinates being held constant. The stresses (2.6) are related to the strains (2.1) by constitutive equations.

Material response functions employed in constitutive relations in the Cosserat theory include dependence on the reference values (2.3) along with $D_3$, and must remain unaltered under reflection of the director,

$$\mathbf{d} \to -\mathbf{d}, \qquad \mathbf{D} \to -\mathbf{D}, \tag{2.8}$$

where **D** is the value of **d** in the reference state. For plates and shells of uniform thickness, it is appropriate to take

$$D_\alpha = 0, \qquad D_3 = 1. \tag{2.9}$$

Hence, the normal components of **D** and **d** become fixed in direction with respect to the surface normal, and invariance under the reflection (2.8) must be replaced by invariance under the transformations

$$d_\alpha \to -d_\alpha, \qquad d_3 \to d_3. \tag{2.10}$$

The transformations (2.10) must be supplemented by reflection of the surface normal with the result that the transformations

$$b_{\alpha\beta} \to -b_{\alpha\beta}, \qquad B_{\alpha\beta} = -B_{\alpha\beta} \tag{2.11}$$

must be considered along with (2.10). As a consequence of (2.10) and (2.11), special material response functions based on the reference values (2.9) must be invariant under the transformations

$$e_{\alpha\beta} \to e_{\alpha\beta}, \qquad \kappa_{\alpha\beta} \to -\kappa_{\alpha\beta}, \qquad \kappa_{3\alpha} \to \kappa_{3\alpha}, \qquad \delta_\alpha \to -\delta_\alpha, \qquad \delta_3 \to \delta_3. \tag{2.12}$$

In order that the mechanical power $P$ remain invariant under the transformations (2.12), the stresses must transform according to

$$\overline{N}^{\alpha\beta} \to \overline{N}^{\alpha\beta}, \quad M^{\alpha\beta} \to -M^{\alpha\beta}, \quad M^{3\alpha} \to M^{3\alpha}, \quad m^\alpha \to -m^\alpha, \quad m^3 \to m^3. \tag{2.13}$$

Material response functions which depend on the stresses (2.6) and the reference values (2.9) must be invariant under the transformations (2.13).

The loading function, or yield function, $f$ plays a central role in constitutive relations for plastic materials. In the general theory, $f$ depends on the stresses (2.6), the plastic strains (2.1) and the temperature. In the following, attention is restricted to isothermal deformations and, in order to identify the loading function with the von Mises yield condition of three-dimensional plasticity, the following quadratic form in the streses is adopted:

$$\begin{aligned} 2f = {} & [\zeta_1 A_{\alpha\beta}A_{\gamma\delta} + \zeta_2(A_{\alpha\gamma}A_{\beta\delta} + A_{\alpha\delta}A_{\beta\gamma})]\overline{N}^{\alpha\beta}\overline{N}^{\gamma\delta} + \zeta_3 A_{\alpha\beta}m^\alpha m^\beta + \zeta_4(m^3)^2 \\ & + (\zeta_5 A_{\alpha\beta}A_{\gamma\delta} + \zeta_6 A_{\alpha\gamma}A_{\beta\delta} + \zeta_7 A_{\alpha\delta}A_{\beta\gamma})M^{\alpha\beta}M^{\gamma\delta} + \zeta_8 A_{\alpha\beta}M^{3\alpha}M^{3\beta} \\ & + 2\zeta_9 A_{\alpha\beta}\overline{N}^{\alpha\beta}m^3, \end{aligned} \tag{2.14}$$

where the coefficients $\zeta_1 \ldots \zeta_9$ are constants. This quadratic form is appropriate for isotropic plates and shells of uniform thickness, and clearly meets the invariance requirements (2.13). A notable consequence of (2.13) is the absence of terms of the form $\overline{N}^{\alpha\beta}M^{\gamma\delta}$. Such terms can appear, however, in yield conditions which are not quadratic; examples of which have been discussed and summarized by Robinson[9, 10]. The various yield conditions considered in [9, 10] do meet the requirement of invariance under the transformations (2.13), with $M^{3\alpha} = m^3 = 0$.

The flow rule introduced here for the plastic strain rates during loading ($\dot{f} > 0$) is based on the commonly assumed normality condition, which yields the following relations:

$$\dot{e}_{\alpha\beta} = \Lambda\frac{\partial f}{\partial \overline{N}^{\alpha\beta}}, \qquad \dot{\kappa}_{\alpha\beta} = \Lambda\frac{\partial f}{\partial M^{\alpha\beta}}, \qquad \dot{\kappa}_{3\alpha} = \Lambda\frac{\partial f}{\partial M^{3\alpha}},$$

$$\dot{\delta}_\alpha = \Lambda\frac{\partial f}{\partial m^\alpha}, \qquad \dot{\delta}_3 = \Lambda\frac{\partial f}{\partial m^3}, \tag{2.15}$$

where $\Lambda = \lambda\dot{f}$, $\dot{f} > 0$.

The scalar factor $\lambda$ is determined by a hardening rule. The value of $f$, at which plastic flow resumes after unloading and subsequent reloading, is commonly related to the plastic work, that is,

$$f = \chi(W),$$

where $W = \int P\,dt$ is the plastic work. Thus, during plastic flow, the additional relations

$$\dot{f} = \frac{d\chi}{dW}P > 0, \qquad P > 0 \tag{2.16}$$

prevail.

The constitutive relations (2.14)–(2.16) meet all the requirements of the general nonlinear theory in [6], and reflect the von Mises yield condition and Lévy–Mises flow rule of three-dimensional plasticity for moderately large plastic deformations. In order to continue to reflect this correspondence with three-dimensional plasticity for arbitrarily large plastic deformations, the relations (2.14)–(2.16) must be modified to include dependence on the plastic strains.

The equilibrium equations complete the specification of field equations for the problem at hand. In the absence of body force and surface loading, they are

$$\begin{gathered}
N^{\alpha\beta}|_{\beta} - b^{\alpha}_{\beta}N^{3\beta} = 0, \qquad N^{3\beta}|_{\beta} + b_{\alpha\beta}N^{\beta\alpha} = 0,\\
M^{\alpha\beta}|_{\beta} - b^{\alpha}_{\beta}M^{3\beta} = m^{\alpha}, \qquad M^{3\beta}|_{\beta} + b_{\alpha\beta}M^{\beta\alpha} = m^{3},\\
N^{3\alpha} + m^{3}d^{\alpha} - m^{\alpha}d^{3} + M^{3\gamma}\lambda^{\alpha}_{.\gamma} - M^{\alpha\gamma}\lambda^{3}_{\gamma} = 0,\\
\overline{N}^{\alpha\beta} = \overline{N}^{\beta\alpha} = N^{\beta\alpha} - m^{\alpha}d^{\beta} - M^{\alpha\gamma}\lambda^{\beta}_{.\gamma} = 0.
\end{gathered} \tag{2.17}$$

The last equation of the set (2.17) can be taken as the equation defining the force resultants $N^{\alpha\beta}$.

It remains to determine $\lambda$ and the nine constitutive coefficients $\zeta_i$ appearing in $f$. Except for $\zeta_3$ and $\zeta_8$, the constitutive coefficients can be determined by considering conditions of incipient yield under simple loadings on a flat rectangular plate.

## 3 DETERMINATION OF THE CONSTITUTIVE COEFFICIENTS FOR A RIGID-PLASTIC COSSERAT SURFACE

The convected coordinates $\theta^{\alpha}$ are now taken to coincide with rectangular Cartesian coordinates $y_1, y_2$ in an undeformed flat rectangular plate, where the $y_1, y_2$ axes of the rectangular Cartesian reference frame $y_k$ ($k = 1, 2, 3$) lie in the middle surface of the plate and are parallel to the sides. In an undeformed flat plate, the strains (2.1) and kinematic variables (2.4) are all zero. Thus it follows from $(2.17)_6$ that there is the equivalence

$$\overline{N}^{\alpha\beta} = N^{\alpha\beta}. \tag{3.1}$$

The plate, as a three-dimensional body, is acted upon by stresses $\sigma_{kl}$, which are the rectangular Cartesian stress components referred to the axes $y_k$. Loading by uniform uniaxial stress $\sigma_{11} = \sigma_0$, where $\sigma_0$ is the yield stress of the material, is considered first. For this loading,

$$\overline{N}^{11} = \sigma_0 h = N_0, \tag{3.2}$$

where $h$ is the thickness of the undeformed plate, and all other stress components (2.6) are zero. In order to have formal correspondence between $f$ and the familiar form of the second invariant of the stress deviator for plane stress, the value of $f$ at incipient yield is related to $N_0$ by

$$f_0 = N_0^2/3. \tag{3.3}$$

When $f$ is evaluated using (2.14) for the loading (3.2) and equated to $f_0$, the result is obtained,

$$\zeta_1 + 2\zeta_2 = 2/3. \tag{3.4}$$

A condition of incipient yield occurs under in-plane shear when $\sigma_{12} = \sigma_0/\sqrt{3}$; that is, when

$$\overline{N}^{12} = N_0/\sqrt{3}. \tag{3.5}$$

When $f$ is evaluated for the loading (3.5), and (3.3) is used, $\zeta_2$ is determined. The result is

$$\zeta_2 = 1/2. \tag{3.6}$$

Then, from (3.4) and (3.6), it follows that

$$\zeta_1 = -1/3. \tag{3.7}$$

Normal loading $\sigma_{33} = \sigma_0$ corresponds to

$$m^3 = N_0 \tag{3.8}$$

in terms of the Cosserat stresses. The loading (3.8) leads to the determination of $\zeta_4$ given by

$$\zeta_4 = 2/3. \tag{3.9}$$

The coefficient $\zeta_9$ can be determined using the condition that the von Mises yield condition for a three-dimensional body is unaffected by hydrostatic stress. The loading function $f$ reflects this aspect of three-dimensional material response provided

$$f = 0, \tag{3.10}$$

when

$$\overline{N}^{11} = \overline{N}^{22} = m^3 \tag{3.11}$$

and all other stress components (2.6) are zero. From (3.6), (3.7) and (3.9) it follows that condition (3.10) is satisfied provided

$$\zeta_9 = -1/3, \tag{3.12}$$

and it can be noted, recalling (3.7), that $\zeta_1 = \zeta_9$.

The coefficients $\zeta_5$, $\zeta_6$, $\zeta_7$ can be determined by considering values of the couple resultants $M^{\alpha\beta}$ which produce a condition of incipient yield. If incipient yield is produced by uniform uniaxial tensile stress $\sigma_{11} = \sigma_0$ acting over one half of the plate thickness where $y_3 > 0$, and by uniform uniaxial compressive stress $\sigma_{11} = -\sigma_0$ acting over the other half where $y_3 < 0$, there is a couple resultant $M^{11}$ given by

$$M^{11} = \sigma_0 h^2/4. \tag{3.13}$$

Evaluating $f$ using the right side of (3.13), with all other stress components (2.6) zero, and equating the result to $f_0$ leads to

$$\zeta_5 + \zeta_6 + \zeta_7 = 32/(3h^2). \tag{3.14}$$

When the couple resultants $M^{\alpha\beta}$ are related to the couple resultants of the stresses $\sigma_{kl}$ in a three-dimensional plate, the symmetry condition $\sigma_{kl} = \sigma_{lk}$ of the Cartesian stresses implies the symmetry condition $M^{\alpha\beta} = M^{\beta\alpha}$. Thus

$$\zeta_6 = \zeta_7. \tag{3.15}$$

Next, yielding under the action of a twisting couple $M^{12}$ is considered. A state of incipient yield is produced by uniform shear stress $\sigma_{12} = \sigma_0/\sqrt{3}$ acting over one half of the plate thickness where $y_3 > 0$, and by uniform stress $\sigma_{12} = -\sigma_0/\sqrt{3}$ acting over the other half where $y_3 < 0$. There is a couple resultant,

$$M^{12} = \sigma_0 h^2/(4\sqrt{3}), \tag{3.16}$$

and all other stress components (2.6) are zero. When $f$ is evaluated using (3.16) and set equal to $f_0$, it is found that

$$\zeta_6 = \zeta_7 = 8/h^2. \tag{3.17}$$

From (3.14) and (3.17), it follows that

$$\zeta_5 = -16/(3h^2). \tag{3.18}$$

Values for the constitutive coefficients $\zeta_i$ were determined in [6] by assuming that the three-dimensional Cartesian stress component $\sigma_{33}$ is uniform through the thickness, while all other components of $\sigma_{kl}$ vary linearly through the thickness. The loading function $f$ was then averaged through the thickness and related to the von Mises yield condition. The coefficients $\zeta_1$, $\zeta_2$, $\zeta_4$, $\zeta_9$ obtained by that procedure then just depend on the average values of stress, and accordingly correspond to the values found here. There is an inconsequential difference by a factor of $h/2$, which is due to assigning different dimensions to $f$, and due to a different matching of $f$ to the condition of yielding under uniform uniaxial stress. There is significant difference, however, in the coefficients $\zeta_5$, $\zeta_6$, $\zeta_7$ which govern bending and twisting. The piecewise uniform three-dimensional stress fields employed here to describe pure bending and twisting satisfy the yield condition pointwise at all values of $y_3$ through the thickness, whereas the linearly varying stresses considered in [6] satisfy the yield condition just in an average sense.

Finally, to determine $\lambda$, the hardening parameter $H$ is introduced as the slope of the stress–strain curve for loading in uniaxial tension or compression, that is, for loading of the plate parallel to the axis $y_1$,

$$\dot{\sigma} = H\dot{e}_{11}, \tag{3.19}$$

where $\sigma = \overline{N}^{11}/h$, and it is recalled that elastic strains are neglected. For continuing deformation described by (3.19), the mechanical power $P$ and the loading function $f$ can be written

$$P = h\sigma\dot{\sigma}/H, \qquad f = \sigma^2 h^2/3. \tag{3.20}$$

When the relations (3.20) are introduced into (2.16), it is found that

$$\frac{d\chi}{dW} = 2Hh/3. \tag{3.21}$$

It is now noted that $f$ is a homogeneous function of degree 2 in the stresses. In view of (2.7) and (2.15), it follows that

$$P = 2\lambda f\dot{f}. \tag{3.22}$$

Comparison of (3.22) with (2.16) yields the relation

$$\frac{d\chi}{dW} = \frac{1}{2\lambda f}. \tag{3.23}$$

Equating the right sides of (3.21) and (3.23) leads to the relation for $\lambda$,

$$\lambda = 3/(4Hhf). \tag{3.24}$$

Thus $\lambda$ and the constitutive coefficients $\zeta_5$, $\zeta_6$, $\zeta_7$ depend on $h$. In the expressions determining these quantities, (3.24), (3.18) and (3.17), respectively, $h$ refers to the thickness of the undeformed plate, and hence is a constant.

## 4 SMALL DEFORMATIONS SUPERPOSED ON UNIFORM FLOW IN A RIGID–PLASTIC COSSERAT PLATE

Deformation of an initially flat plate is described by referring material points $\theta^\alpha$ on the middle surface to the fixed rectangular Cartesian axes $y_k$, introduced in Section 3, by the relations

$$y_1 = k_1\theta^1 + \epsilon u_1, \qquad y_2 = k_2\theta^2, \qquad y_3 = \epsilon w, \tag{4.1}$$

where $k_1 = k_1(t)$ and $k_2 = k_2(t)$ are extension ratios which describe uniform extension, and

$$u_1 = u_1(\theta^1, t), \qquad w = w(\theta^1, t)$$

represent superposed deformations. To complete the description of the deformation, the components of the director $\mathbf{d}$ are written

$$d_1 = \epsilon\delta_1', \qquad d_2 = 0, \qquad d_3 = 1 + \Delta_3 + \epsilon\delta_3', \tag{4.2}$$

where

$$\Delta_3 = \Delta_3(t)$$

describes uniform thickness change due to uniform extension, and

$$\delta_1' = \delta_1'(\theta^1, t), \qquad \delta_3' = \delta_3'(\theta^1, t)$$

describe superposed deformations. In eqns (4.1) and (4.2), $\epsilon$ is an arbitrarily small positive constant which identifies terms which vanish with the superposed deformations. Throughout what follows, all terms in $\epsilon^2$ and smaller are neglected.

For the surface defined by (4.1), the components of the covariant surface metric tensor $a_{\alpha\beta}$ are given by

$$a_{11} = k_1^2 + \epsilon\cdot 2k_1u_{1,1}, \qquad a_{22} = k_2^2, \qquad a_{12} = a_{21} = 0 \tag{4.3}$$

and the components of its conjugate $a^{\alpha\beta}$ by

$$a^{11} = k_1^{-2}(1 - \epsilon\cdot 2k_1^{-1}u_{1,1}), \qquad a^{22} = k_2^{-2}, \qquad a^{12} = a^{21} = 0. \tag{4.4}$$

The coefficients $b_{\alpha\beta}$ in the second fundamental form are

$$b_{11} = \epsilon w_{,11}, \qquad b_{12} = b_{21} = b_{22} = 0. \tag{4.5}$$

In view of (4.3), the only non-zero Christoffel symbols are $\Gamma_{111}$ and $\Gamma^1_{11}$, and they have the values

$$\Gamma_{111} = \epsilon k_1 u_{1,11}, \qquad \Gamma^1_{11} = \epsilon k_1^{-1} u_{1,11} \tag{4.6}$$

Hence, surface covariant differentiation of $\epsilon$-dependent quantities reduces to partial differentiation, when terms in $\epsilon^2$ are dropped. Thus the kinematic variables $\lambda_{\alpha\beta}$, $\lambda_{3\alpha}$ have the form

$$\begin{aligned} &\lambda_{11} = \epsilon(\delta'_{1,1} - (1 + \Delta_3)w_{,11}), \qquad \lambda_{12} = \lambda_{21} = \lambda_{22} = 0, \\ &\lambda_{31} = \epsilon\delta'_{3,1}, \qquad \lambda_{32} = 0. \end{aligned} \tag{4.7}$$

From (4.3)–(4.7), the strain rates are found to be

$$\begin{aligned} &\dot{e}_{11} = k_1\dot{k}_1 + \epsilon\,(\dot{\overline{k_1 u_{1,1}}}), \qquad \dot{e}_{22} = k_2\dot{k}_2, \qquad \dot{e}_{12} = \dot{e}_{21} = 0, \\ &\dot{\kappa}_{11} = \epsilon(\dot{\delta}'_{1,1} - \dot{\Delta}_3 w_{,11} - (1 + \Delta_3)\dot{w}_{,11}), \qquad \dot{\kappa}_{12} = \dot{\kappa}_{21} = \dot{\kappa}_{22} = 0, \\ &\dot{\kappa}_{31} = \epsilon\dot{\delta}'_{3,1}, \qquad \dot{\kappa}_{32} = 0, \\ &\dot{\delta}_1 = \epsilon\dot{\delta}'_1, \qquad \dot{\delta}_2 = 0, \qquad \dot{\delta}_3 = \dot{\Delta}_3 + \epsilon\dot{\delta}'_3. \end{aligned} \tag{4.8}$$

Uniform extension of the plate is produced by uniform axial load $\overline{N}^{11} = N$ acting parallel to the $y_1$ axis. When the perturbation is superposed, the appropriate expressions for the stresses to correspond to the strain rates (4.8) are

$$\begin{aligned} &\overline{N}^{11} = N + \epsilon\overline{N}'^{11}, \qquad \overline{N}^{22} = \epsilon\overline{N}'^{22}, \qquad \overline{N}^{12} = \overline{N}^{21} = 0, \\ &M^{11} = \epsilon M'^{11}, \qquad M^{22} = \epsilon M'^{22}, \qquad M^{12} = M^{21} = 0, \\ &M^{31} = \epsilon M'^{31}, \qquad M^{32} = 0, \\ &m^1 = \epsilon m'^1, \qquad m^2 = 0, \qquad m^3 = \epsilon m'^3. \end{aligned} \tag{4.9}$$

The plate is considered to be perfectly flat in the undeformed state, and to undergo uniform extension for $0 < t < t_1$. Then, at time $t = t_1$, a perturbation is introduced by means of impulsive loads applied to the plate surfaces $y_3 = \pm h_1/2$, where $h_1$ is the plate thickness at time $t_1$. As a result, for $t > t_1$, the deformation is described by (4.1), (4.2) and the stress field by (4.9). In the undeformed plate, the material coordinates $\theta^\alpha$ can be chosen so that $k_1(0) = k_2(0) = 1$. Hence, the components of $A_{\alpha\beta}$ and its conjugate $A^{\alpha\beta}$ are simply

$$A_{11} = A_{22} = A^{11} = A^{22} = 1, \qquad A_{12} = A_{21} = A^{12} = A^{21} = 0. \tag{4.10}$$

For the reference values (4.10), the stresses (4.9), and the values (3.6), (3.7), (3.9), (3.12) for the constitutive coefficients, $\lambda$ and $\dot{f}$ are determined as

$$\lambda = \frac{9}{4HhN^2}[1 - \epsilon(2\overline{N}'^{11} - \overline{N}'^{22} - m'^3)/N], \tag{4.11}$$

$$\dot{f} = \tfrac{4}{9}N\dot{N} + \frac{\epsilon}{3}[N(2\dot{\overline{N}}'^{11} - \dot{\overline{N}}'^{22} - \dot{m}'^3) + \dot{N}(2\overline{N}'^{11} - \overline{N}'^{22} - m'^3)]. \tag{4.12}$$

In quasi-static deformation of elastic–plastic or rigid–plastic bodies, there is no natural time, with the result that the time scale can be chosen arbitrarily. The expression for $\dot{f}$ can be simplified by choosing the loading rate, such that

$$N = N_1\,e^{(t - t_1)}, \tag{4.13}$$

where $N_1$ is the value of $N$ at time $t_1$. Then $\dot{N} = N$, and the expression for $\dot{f}$ can be rewritten as

$$\dot{f} = \tfrac{2}{3}N^2 + \epsilon \frac{N}{3}[(2\dot{\bar{N}}'^{11} - \dot{\bar{N}}'^{22} - m'^3) + (2\bar{N}'^{11} - \bar{N}'^{22} - m'^3)]. \tag{4.14}$$

The flow rule (2.15) is evaluated for the stresses (4.9) and the reference values (4.10) using the values for the constitutive coefficients (3.6), (3.7), (3.9), (3.12), (3.17) and (3.18), and using expressions (4.11) and (4.14) for $\lambda$ and $\dot{f}$, respectively. The nontrivial relations are

$$\begin{aligned}
Hh\dot{e}_{11} &= N + \epsilon(\dot{\bar{N}}'^{11} - \tfrac{1}{2}\dot{\bar{N}}'^{22} - \tfrac{1}{2}\dot{m}'^3),\\
2Hh\dot{e}_{22} &= -N - \epsilon[(\dot{\bar{N}}'^{11} - \tfrac{1}{2}\dot{\bar{N}}'^{22} - \tfrac{1}{2}\dot{m}'^3) - \tfrac{3}{2}(\bar{N}'^{22} - m'^3)],\\
2Hh\dot{\delta}_3 &= -N - \epsilon[(\dot{\bar{N}}'^{11} - \tfrac{1}{2}\dot{\bar{N}}'^{22} - \tfrac{1}{2}\dot{m}'^3) + \tfrac{3}{2}(\bar{N}'^{22} - m'^3)],\\
Hh^3\dot{\kappa}_{11} &= \epsilon\cdot 8(2M'^{11} - M'^{22}),\\
Hh^3\dot{\kappa}_{22} &= \epsilon\cdot 8(2M'^{22} - M'^{11}),\\
2Hh\dot{\kappa}_{31} &= \epsilon\cdot 3\zeta_8 M'^{31},\\
2Hh\dot{\delta}_1 &= \epsilon\cdot 3\zeta_3 m'^1.
\end{aligned} \tag{4.15}$$

The relations which describe uniform extension of the plate are obtained by matching corresponding $\epsilon$-independent terms in the kinematic relations (4.8) and the constitutive relations (4.15). They are

$$Hhk_1\dot{k}_1 = N, \qquad 2Hhk_2\dot{k}_2 = -N, \qquad 2Hh\dot{\Delta}_3 = -N. \tag{4.16}$$

The relations which describe superposed deformations are obtained by matching corresponding $\epsilon$-dependent terms in the two sets of equations, (4.8) and (4.15).

The equilibrium conditions for continuing uniform extension of the plate are satisfied trivially. For the superposed deformations included in (4.1) and (4.2), the associated stresses (4.9) must satisfy five nontrivial equilibrium equations which follow from (2.17) and (4.3)–(4.7). They are

$$\begin{aligned}
&\bar{N}'^{11}{}_{,1} + 2k_1^{-1}Nu_{1,11} = 0, \qquad N'^{31}{}_{,1} + Nw_{,11} = 0,\\
&M'^{11}{}_{,1} = m'^1, \qquad M'^{31}{}_{,1} = m'^3, \qquad N'^{31} - m'^1(1 + \Delta_3), = 0.
\end{aligned} \tag{4.17}$$

The equations governing superposed deformations, that is, the kinematic relations (4.8), the constitutive relations (4.15) and the equilibrium equations (4.17), uncouple into two sets. One set yields an eigensolution for bending instability in a uniformly compressed plate, while the other set yields an eigensolution for symmetric bulging or thinning.

## 5. EIGENSOLUTION FOR BENDING INSTABILITY IN A COMPRESSED COSSERAT PLATE

The transverse shear $N'^{31}$ can be eliminated between the two equilibrium equations $(4.17)_2$ and $(4.17)_5$, with the result that

$$(1 + \Delta_3)m'^1{}_{,1} + Nw_{,11} = 0. \tag{5.1}$$

In view of the fact that $\dot{\kappa}_{22} = 0$, it follows from $(4.15)_5$ that

$$M'^{22} = M'^{11}/2, \tag{5.2}$$

and the constitutive relation for $\dot{\kappa}_{11}$ can be rewritten as

$$(Hh^3/12)\dot{\kappa}^{11} = \epsilon M'^{11}. \quad (5.3)$$

The equilibrium conditions $(4.17)_3$ and (5.1), along with the kinematic relation for $\dot{\kappa}_{11}$ in the set (4.8), and the constitutive relations $(4.15)_7$ and (5.3) comprise a system of four equations for the four independent unknowns $m'^1$, $M'^{11}$, $w$, $\delta'_1$. It is convenient to eliminate $m'^1$ and $M'^{11}$, and thereby obtain two simultaneous equations for $w$ and $\delta'_1$. From $(4.8)_5$, $(4.16)_3$, and (5.3), it follows that

$$M'^{11} = (Hh^3/12)[\delta'_{1,1} + (NH^{-1}h^{-1}/2)w_{,11} - (1 + \Delta_3)\dot{w}_{,11}]. \quad (5.4)$$

Then from $(4.15)_7$, $(4.17)_3$ and (5.4), the result

$$\delta'_{1,11} - (8\zeta_3^{-1}h^{-2})\delta'_1 + (NH^{-1}h^{-1}/2)w_{,111} - (1 + \Delta_3)\dot{w}_{,111} = 0 \quad (5.5)$$

is obtained. Combining $(4.15)_7$ and (5.1) yields

$$(1 + \Delta_3)\dot{\delta}'_{1,1} + (3\zeta_3 NH^{-1}h^{-1}/2)w_{,11} = 0. \quad (5.6)$$

In the coefficients in (5.5) and (5.6), $N$, $H$ and $\Delta_3$ are all time-dependent quantities. The hardening modulus $H$ can be considered to depend on the uniform uniaxial stress $\sigma = N/h$, where $\sigma \propto e^t$. The coefficients can be simplified by introducing a new time scale through a new independent variable $\tau$ which is the increment, over the small time interval $t - t_1 \geqslant 0$, in the uniform part of the axial strain $e_{11}$; that is,

$$\tau = -(N - N_1)/(Hh). \quad (5.7)$$

In eqn (5.7), $N \propto e^t$, while $H$ is held constant at its value at time $t_1$. Thus the actual stress–strain curve is replaced by the tangent to the curve at the point where $t = t_1$. The variable $\tau$ is a nonnegative monotonically increasing quantity for loading in compression, $N < 0$. For small $\tau$, a solution to eqn (5.5), (5.6) for sinusoidal buckling can be taken as

$$\delta'_1 = A\, e^{p\tau} \sin(\pi\theta^1/l), \qquad w = B\, e^{p\tau} \cos(\pi\theta^1/l), \quad (5.8)$$

where $l$ is the half wavelength, and $A$, $B$ and $p$ are undetermined constants. When substitutions are made in (5.5), (5.6) using (5.8), $NH^{-1}$ becomes a common factor in all coefficients, and cancels from the equations. If the deformation associated with uniform extension is considered to be only moderately large, then $\Delta_3$ can be neglected compared to unity, and (5.5) and (5.6) reduce to equations with constant coefficients. In order that $A$ and $B$ not vanish, $p$ must satisfy the eigencondition

$$p = (3\zeta_3 - 1)/2 + 12l^2/(\pi^2h^2). \quad (5.9)$$

For instability, $p > 0$. For thin plates, $l/h \gg 1$, thus $p \gg 1$, and the shear coefficient $\zeta_3$ has little effect on instability. For thick plates or higher buckling modes in thin plates, $l/h \ll 1$, and the value of $p$ is dominated by $\zeta_3$. The eigencondition (5.9) is compared later with the eigencondition for a three-dimensional plate, and a value for $\zeta_3$ determined.

## 6. EIGENSOLUTION FOR BULGING INSTABILITY IN A COMPRESSED COSSERAT PLATE

For the bulging mode, a single equation in the kinematic variable $\delta'_3$ can easily be obtained. When the kinematic relation for $\dot{\kappa}_{31}$ in (4.8) is used along with the constitutive relation $(4.15)_6$ and the equilibrium equation $(4.17)_4$, the result

$$m'^3 = (2Hh\zeta_8^{-1}/3)\dot{\delta}'_{3,11} \tag{6.1}$$

is obtained. Next, the constitutive equation $(4.15)_1$ is multiplied through by two, and corresponding sides of $(4.15)_2$ and $(4.15)_3$ are added to the result. It is found that

$$\dot{e}_{11} + \dot{e}_{22} + \delta_3 = 0. \tag{6.2}$$

Substitutions are made in eqn (6.2) using the kinematic relations (4.8). Partitioning the result into $\epsilon$-independent and $\epsilon$-dependent parts yields the two equations

$$k_1\dot{k}_1 + k_2\dot{k}_2 + \dot{\Delta}_3 = 0, \qquad \overline{(k_1 u_{1,1})}^{\boldsymbol{\cdot}} + \dot{\delta}'_3 = 0. \tag{6.3}$$

Equation $(6.3)_2$ can be integrated with respect to $t$ to obtain

$$k_1 u_{1,1} + \delta'_3 = 0. \tag{6.4}$$

An arbitrary function of $\theta^1$ could be included in eqn (6.4), but this function is zero because $u_1$ and $\delta'_3$ vanish simultaneously at time $t_1$. From $(4.8)_1$, $(4.15)_1$ and $(6.3)_2$ it can be noted that

$$-Hh\dot{\delta}'_3 = \dot{\overline{N}}'^{11} - \tfrac{1}{2}\dot{\overline{N}}'^{22} - \tfrac{1}{2}\dot{m}'^3. \tag{6.5}$$

With $H$ held constant, as considered in Section 5, eqn (6.5) can be integrated with respect to $t$, and yields

$$-Hh\delta'_3 = \overline{N}'^{11} - \tfrac{1}{2}\overline{N}'^{22} - \tfrac{1}{2}m'^3. \tag{6.6}$$

An arbitrary function of $\theta^1$ could be included in eqn (6.6), and this function related to the conditions of impulsive loading at time $t_1$. However, for present purposes, only the homogeneous solution of the final differential equation is of interest, so this arbitrary function of $\theta^1$ is omitted.

Inasmuch as the kinematic relation for $\dot{e}_{22}$ in (4.8) has no $\epsilon$-dependent term, the $\epsilon$-dependent term in the constitutive relation $(4.15)_2$ for $\dot{e}_{22}$ must vanish; that is,

$$\dot{\overline{N}}'^{11} - \tfrac{1}{2}\dot{\overline{N}}'^{22} - \tfrac{1}{2}\dot{m}'^3 - \tfrac{3}{2}(\overline{N}'^{22} - m'^3) = 0. \tag{6.7}$$

Then eqns (6.1), (6.5) and (6.7) can be used to solve for $\overline{N}'^{22}$. The result is

$$\overline{N}'^{22} = (2Hh/3)[\zeta_8^{-1}\dot{\delta}'_{3,11} - \dot{\delta}'_3]. \tag{6.8}$$

When substitutions are made in (6.6) from (6.1) and (6.8), $\overline{N}'^{11}$ is determined as

$$\overline{N}'^{11} = (Hh/3)[2\zeta_8^{-1}\dot{\delta}'_{3,11} - \dot{\delta}'_3 - 3\delta'_3]. \tag{6.9}$$

Finally, substitutions from eqns (6.4) and (6.9) in the equilibrium equation $(4.17)_1$ lead to

$$2\zeta_8^{-1}\dot{\delta}'_{3,111} - \dot{\delta}'_{3,1} - 3(1 + 2Nk_1^{-2}H^{-1}h^{-1})\delta'_{3,1} = 0. \tag{6.10}$$

During the short time interval that the growth of the perturbation is followed, the value of $N$ in the coefficient of $\delta'_{3,1}$ in eqn (6.10) can be taken as constant. Also, for deformation due to uniform extension which is only moderately large, $k_1$ can be taken as unity. Thus if a solution to eqn (6.10) is written

$$\delta'_3 = A\,e^{rt}\sin(\pi\theta^1/l), \tag{6.11}$$

where $A$ and $r$ are undetermined constants, and $l$ is the half wavelength in a sinusoidal buckling mode, $r$ must satisfy the condition

$$r = -\frac{3(1 + 2NH^{-1}h^{-1})}{1 + 2\pi^2 h^2/(\beta l^2)} \tag{6.12}$$

in order that $A \neq 0$. In (6.12), $\beta$ is a positive dimensionless coefficient, such that

$$\zeta_8 = \beta h^{-2}. \tag{6.13}$$

Also, in eqn (6.12), $N$ and $H$ are understood to be the values at time $t_1$. For instability, $r > 0$. Thus there is no solution for sinusoidal necking instability in tension. There is instability which appears as symmetric sinusoidal bulging or thinning in a compressed plate, when

$$2\,|N|\,H^{-1}h^{-1} > 1, \qquad N < 0. \tag{6.14}$$

## 7. SMALL DEFORMATIONS SUPERPOSED ON UNIFORM FLOW IN A RIGID–PLASTIC THREE-DIMENSIONAL PLATE

The general theory of an elastic–plastic continuum due to Green and Naghdi[7] is now applied to obtain governing equations for small plane-strain deformations superposed on continuing uniform flow produced in a rectangular rigid–plastic plate by uniaxial compression. Fixed rectangular Cartesian axes $x_k$ $(k = 1, 2, 3)$ are introduced with the $x_1$ and $x_3$ axes lying in the middle surface of the undeformed plate parallel to the sides. Uniform compressive load acts in the direction of the $x_1$ axis. Material coordinates are denoted by $X_K$ $(K = 1, 2, 3)$ and are related to $x_k$ by

$$x_1 = \lambda_1 X_1 + \epsilon\lambda_1^{-1}w_1, \qquad x_2 = \lambda_2 X_2 + \epsilon\lambda_2^{-1}w_2, \qquad x_3 = \lambda_3 X_3, \tag{7.1}$$

where $\lambda_K = \lambda_K(t)$ are extension ratios which describe continuing uniform extension, and

$$w_1 = w_1(X_1, X_2, t), \qquad w_2 = w_2(X_1, X_2, t)$$

describe superposed deformations. In the undeformed plate, the material coordinates $X_K$ coincide with the fixed rectangular Cartesian coordinates $x_k$. Thus $\lambda_K = 1$ at time $t = 0$ at the start of deformation. As considered for the Cosserat plate, deformation of the three-dimensional plate consists of uniform extension for $0 < t < t_1$. At time $t_1$, the plate surfaces $X_2 = \pm h/2$ are loaded impulsively with the result that, for $t > t_1$, the continuing uniform flow is perturbed. In eqn (7.1), $\epsilon$ is an arbitrarily small positive constant. Subsequently, all quantities and all relations are linearized with respect to $\epsilon$.

In the general theory[7], nonsymmetric stress components $\Pi_{Kk}$ are defined with respect to unit area in the undeformed body. Surface traction $p_k$ is related to these components by

$$p_k = N_K \Pi_{Kk}, \tag{7.2}$$

where $N_K$ is the outward unit normal, in the undeformed state, of the area element on which $p_k$ acts. When the body force is zero, the differential equations of equilibrium are

$$\Pi_{Kk,K} = 0. \tag{7.3}$$

Symmetric stress components $S_{KL}$, which are employed in constitutive relations, are related to $\Pi_{Kk}$ by

$$\Pi_{Kk} = x_{k,L}S_{LK}. \tag{7.4}$$

The stress field associated with the perturbed uniform flow described by eqn (7.1) is written

$$S_{KL} = \begin{bmatrix} -P + \epsilon s_{11} & \epsilon s_{12} & 0 \\ \epsilon s_{12} & \epsilon s_{22} & 0 \\ 0 & 0 & \epsilon s_{33} \end{bmatrix}, \tag{7.5}$$

where $P = P(t)$ denotes uniform compressive stress, and

$$s_{KL} = s_{KL}(X_1, X_2, t)$$

are superposed stresses. From eqns (7.1) and (7.3)–(7.5), it follows that there are two nontrivial equilibrium equations, which are

$$\begin{aligned} \lambda_1^2(s_{11,1} + s_{12,2}) - Pw_{1,11} &= 0, \\ \lambda_2^2(s_{12,1} + s_{22,2}) - Pw_{2,11} &= 0. \end{aligned} \tag{7.6}$$

The plate surfaces $X_2 = \pm h/2$ are free from applied tractions, provided

$$s_{12} = s_{22} = 0, \qquad X_2 = \pm h/2, \tag{7.7}$$

which follow from eqns (7.1), (7.2) and (7.4). The strain components $e_{KL}$, which are used in the constitutive relations along with the stress components $S_{KL}$, are specified by

$$2e_{KL} = x_{k,K}x_{k,L} - \delta_{KL}, \tag{7.8}$$

where $\delta_{KL}$ is the Kronecker delta. The nonzero strain components for the deformation described by (7.1) are

$$\begin{aligned} e_{11} &= (\lambda_1^2 - 1)/2 + \epsilon w_{1,1}, \\ e_{22} &= (\lambda_2^2 - 1)/2 + \epsilon w_{2,2}, \\ e_{33} &= (\lambda_3^2 - 1)/2, \\ 2e_{12} &= \epsilon(w_{1,2} + w_{2,1}). \end{aligned} \tag{7.9}$$

Constitutive equations, linearized in $\epsilon$, are now determined for the deformations (7.1) and the stress field (7.5), based on the von Mises yield condition and the Lévy–Mises flow rule. Again, the material is idealized at the outset as rigid–plastic, with the result that the plastic strains coincide with the total strains $e_{KL}$ in eqn (7.8). In the general theory[7], the loading function $f$ depends on the stresses $S_{KL}$, the plastic strains $e_{KL}$ and temperature. The loading function adopted here is the quadratic expression in the stress components $S_{KL}$,

$$2f = (\delta_{KM}\delta_{LN} - \tfrac{1}{3}\delta_{KL}\delta_{MN})S_{MN}S_{KL}, \tag{7.10}$$

which corresponds formally to the von Mises yield condition for isothermal deformations. The constitutive relations for the plastic strain rates $\dot{e}_{KL}$ are then given by

$$\dot{e}_{KL} = \Lambda \frac{\partial f}{\partial S_{KL}}, \tag{7.11}$$

where $\Lambda = \lambda \dot{f}$, $\dot{f} > 0$ and $\lambda = 3/(4Hf)$. The relations (7.10) and (7.11) are the three-dimensional counterparts of (2.14), (2.15) and (3.24). In both sets of relations, $H$ is the slope of the stress–strain curve in uniaxial tension or compression.

Expressions for $\lambda$ and $\dot{f}$, for the state of stress (7.5), are

$$\lambda = 9(4HP^2)^{-1}[1 + \epsilon(2s_{11} - s_{22} - s_{33})/P], \tag{7.12}$$

$$\dot{f} = (2P/3)[\dot{P} - \epsilon(s_{11} - \tfrac{1}{2}s_{22} - \tfrac{1}{2}s_{33})\dot{P}/P - \epsilon(\dot{s}_{11} - \tfrac{1}{2}\dot{s}_{22} - \tfrac{1}{2}\dot{s}_{33})]. \tag{7.13}$$

It is convenient to specify the loading rate by putting

$$P = P_1\, e^{(t-t_1)},$$

where $P_1$ is the value of $P$ at time $t_1$. Then $\dot{P} = P$, and $\dot{f}$ can be rewritten as

$$\dot{f} = (2P^2)/3)[1 - \epsilon(s_{11} - \tfrac{1}{2}s_{22} - \tfrac{1}{2}s_{33}) - \epsilon(\dot{s}_{11} - \tfrac{1}{2}\dot{s}_{22} - \tfrac{1}{2}\dot{s}_{33})]. \tag{7.14}$$

When the flow rule (7.11) is evaluated using eqns (7.5), (7.12) and (7.14), the following constitutive relations result:

$$\begin{aligned} H\dot{e}_{11} &= -P + \epsilon(\dot{s}_{11} - \tfrac{1}{2}\dot{s}_{22} - \tfrac{1}{2}\dot{s}_{33}), \\ 2H\dot{e}_{22} &= P - \epsilon[(\dot{s}_{11} - \tfrac{1}{2}\dot{s}_{22} - \tfrac{1}{2}\dot{s}_{33}) - \tfrac{3}{2}(s_{22} - s_{33})], \\ 2H\dot{e}_{33} &= P - \epsilon[(\dot{s}_{11} - \tfrac{1}{2}\dot{s}_{22} - \tfrac{1}{2}\dot{s}_{33}) + \tfrac{3}{2}(s_{22} - s_{33})], \\ 2H\dot{e}_{12} &= \epsilon \cdot 3s_{12}. \end{aligned} \tag{7.15}$$

It can be noted from eqn (7.15) that

$$\dot{e}_{11} + \dot{e}_{22} + \dot{e}_{33} = 0. \tag{7.16}$$

A description of the deformation due to uniform flow is obtained by matching $\epsilon$-independent terms in the expressions for $\dot{e}_{11}$, $\dot{e}_{22}$ and $\dot{e}_{33}$ obtained from the kinematic relations (7.9) with those in the constitutive relations (7.15). The relations are

$$H\lambda_1\dot{\lambda}_1 = -P, \qquad 2H\lambda_2\dot{\lambda}_2 = P, \qquad 2H\lambda_3\dot{\lambda}_3 = P. \tag{7.17}$$

Equations describing the superposed deformations are obtained by matching $\epsilon$-dependent terms in the expressions for strain rate obtained from eqn (7.9) with those in eqn (7.15).

## 8. EIGENSOLUTIONS FOR BENDING AND BULGING INSTABILITIES IN A COMPRESSED THREE-DIMENSIONAL PLATE

While the equations governing superposed deformations in a uniformly compressed Cosserat plate uncouple into two sets, one set governing bending instability and the other bulging instability, both deformation modes in a three-dimensional plate are governed by the same set of equations. Bending and bulging modes, in a three-dimensional plate, correspond to solutions in which $w_2$ is an even or an odd function, respectively, in the coordinate $X_2$. To proceed with constructing a solution, it is convenient to start with the kinematic relation for the strain rate component $\dot{e}_{11}$ obtained from eqn (7.9), and equate the $\epsilon$-dependent term to the $\epsilon$-dependent term in the constitutive relation $(7.15)_1$. Thus

$$H\dot{w}_{1,1} = \dot{s}_{11} - \tfrac{1}{2}\dot{s}_{22} - \tfrac{1}{2}\dot{s}_{33}. \tag{8.1}$$

As before, $H$ is treated as constant over the small time interval $t - t_1 \geq 0$ that the growth of the perturbation is followed. Integration of (8.1) with respect to $t$ yields

$$Hw_{1,1} = s_{11} - \tfrac{1}{2}s_{22} - \tfrac{1}{2}s_{33}. \tag{8.2}$$

An arbitrary function of $X_1$, $X_2$, which could be related to the impulsive loads at time $t_1$, is omitted because only the homogeneous solution of the resulting differential equations is of interest. Since the strain component $e_{33}$ has no $\epsilon$-dependent term in eqn (7.9), the $\epsilon$-dependent term in the constitutive equation for $\dot{e}_{33}$ is set equal to zero. From $(7.15)_3$ and (8.1), it follows that

$$s_{33} = s_{22} + (2H/3)\dot{w}_{1,1}, \tag{8.3}$$

and substitution for $s_{33}$ from (8.3) in (8.2) yields

$$s_{22} = s_{11} - H(w_{1,1} + \dot{w}_{1,1}/3). \tag{8.4}$$

For reference later, the constitutive equation $(7.15)_4$ for the shear stress $s_{12}$ is written in terms of $w_1$, $w_2$ as

$$s_{12} = (H/3)(\dot{w}_{1,2} + \dot{w}_{2,1}), \tag{8.5}$$

which follows when the kinematic relation $(7.9)_4$ is used.

A relation between $w_1$ and $w_2$ can be obtained using eqn (7.16) and the kinematic relations (7.9). Substitutions are made from (7.9) and (7.16), and the result partitioned into $\epsilon$-dependent and $\epsilon$-independent terms. Setting the $\epsilon$-dependent term to zero yields

$$\dot{w}_{1,1} + \dot{w}_{2,2} = 0. \tag{8.6}$$

Integration of (8.6) with respect to $t$ leads to the result

$$w_{1,1} + w_{2,2} = 0. \tag{8.7}$$

No arbitrary function of $X_1$, $X_2$ is included in the integral (8.7), because $w_1$ and $w_2$ vanish simultaneously at time $t_1$.

Equations (8.4), (8.5) and (8.7), along with the two equations of equilibrium (7.6), comprise a system of five simultaneous equations for the five unknowns $s_{11}$, $s_{22}$, $s_{12}$, $w_1$ and $w_2$. These equations are now solved for the special case of sinusoidal buckling modes. A new independent variable $\tau$ is introduced by putting

$$\tau = (P - P_1)]H, \tag{8.8}$$

which, as in eqn (5.7), is the increment in the uniform compressive strain. In eqn (8.8), $P \propto e'$, while $H$ has a constant value corresponding to the stress $P_1$. Hence,

$$\dot{\tau} = P/H. \tag{8.9}$$

In eqn (8.9), $P$ is now held constant at the value $P_1$. In the equilibrium equations (7.6), the coefficients $\lambda_1^2$, $\lambda_2^2$ are replaced by unity, and $P$ is held constant at the value $P_1$. As a result, the eqns (7.6), (8.4), (8.5) and (8.7) have constant coefficients, and their solution can easily be constructed. For the bending mode, the solution can be written

$$
\begin{aligned}
w_1 &= [k_1 W_1 \sinh(k_1 y) + k_2 W_2 \sinh(k_2 y)]\, e^{p\tau} \cos x,\\
w_2 &= [W_1 \cosh(k_1 y) + W_2 \cosh(k_2 y)]\, e^{p\tau} \sin x,\\
s_{11} &= -\frac{\pi}{l}\{[1 + p(k_1^2 + 1)/3]k_1 W_1 \sinh(k_1 y)\\
&\quad + [1 + p(k_2^2 + 1)/3]k_2 W_2 \sinh(k_2 y)\}P\, e^{p\tau} \sin x, \qquad (8.10)\\
s_{22} &= -\frac{\pi}{l}\{[1 - p(k_1^2 + 1)/3]k_1^{-1} W_1 \sinh(k_1 y)\\
&\quad + [1 - p(k_2^2 + 1)/3]k_2^{-1} W_2 \sinh(k_2 y)\}P\, e^{p\tau} \sin x,\\
s_{12} &= \frac{\pi}{l}\{[p(k_1^2 + 1)/3] W_1 \cosh(k_1 y) + [p(k_2^2 + 1)/3] W_2 \cosh(k_2 y)\}P\, e^{p\tau} \cos x,
\end{aligned}
$$

where

$$x = \pi X_1/l, \qquad y = \pi X_2/l,$$

and $k_1^2$, $k_2^2$ are roots of the second-order equation in $k^2$,

$$pk^4 + [p - 3(H/P - 1)]k^2 + (p - 3) = 0, \qquad (8.11)$$

and $W_1$, $W_2$ are arbitrary constants. The boundary conditions (7.7), $s_{12} = s_{22} = 0$ along $X_2 = \pm h/2$, can be satisfied for $W_1, W_2 \neq 0$, provided

$$\frac{1 - 3p^{-1}(k_2^2 + 1)^{-1}}{1 - 3p^{-1}(k_1^2 + 1)^{-1}} = \frac{k_2}{k_1} \cdot \frac{\tanh[(k_1 \pi h)/(2l)]}{\tanh[(k_2 \pi h)/(2l)]}. \qquad (8.12)$$

For the bulging mode, the functions sinh and cosh are interchanged in eqn (8.10), and the eigencondition becomes

$$\frac{1 - 3p^{-1}(k_2^2 + 1)^{-1}}{1 - 3p^{-1}(k_1^2 + 1)^{-1}} = \frac{k_2}{k_1} \cdot \frac{\tanh[(k_2 \pi h)/(2l)]}{\tanh[(k_1 \pi h)/(2l)]}. \qquad (8.13)$$

## 9 COMPARISON OF EIGENCONDITIONS FOR A COSSERAT PLATE AND A THREE-DIMENSIONAL PLATE

The eigenconditions (5.9) and (6.12) for a Cosserat plate, and (8.12) and (8.13) for a three-dimensional plate, determine the growth rate $p$ of a perturbation as a function of $h/l$. Except for (5.9), which governs the bending mode in a compressed Cosserat plate, these eigenconditions depend on $H/P$. For the common ductile metals with stress–strain curves which have a continuously turning tangent, $H$ decreases rather sharply as $P$ increases, and the range of values $50 > H/P > 5$ would be representative of these materials.

Some general features of the eigenconditions (8.12) and (8.13) for a three-dimensional plate are established first, for $p$ real and positive. The roots $k_1^2$, $k_2^2$ of (8.11) are now specified as

$$k_1^2 = [-b + (b^2 - 4pc)^{1/2}]/(2p), \qquad k_2^2 = [-b - (b^2 - 4pc)^{1/2}]/(2p). \qquad (9.1)$$

where

$$b = p - 3(H/P - 1), \qquad c = p - 3.$$

The roots $k_1^2$, $k_2^2$ are coincident when $p = p^*$, where

$$p^* \approx H/P. \tag{9.2}$$

For $p > p^*$, the roots are complex, and $k_1$, $k_2$ can be taken as a complex conjugate pair with positive real parts. For $3 < p < p^*$, $k_1^2$ and $k_2^2$ are positive and distinct; thus there are positive and distinct values for $k_1$, $k_2$. When $p = 3$, $k_2 = 0$. For $0 < p < 3$, $k_1^2 > 0$, while $k_2^2 < 0$. Thus $k_1$ can be taken as real and positive, while $k_2$ is pure imaginary. For $p > 3$, since $k_1$, $k_2$ then have positive real parts, the two eigenconditions (8.10) and (8.11) coalesce in the limit as $h/l \to \infty$, and have the form

$$\frac{1 - 3p^{-1}(k_2^2 + 1)^{-1}}{1 - 3p^{-1}(k_1^2 + 1)^{-1}} = \frac{k_2}{k_1}. \tag{9.3}$$

This result governs localized surface instability in a thick plate. The eigencondition (9.3) is satisfied in the limit as $p \to 3$, when the numerators on both sides vanish. Numerical computation shows that the eigencondition (8.12) for the bending mode has solutions for $p > 3$, with $h/l$ decreasing as $p$ increases, while the eigencondition (8.13) for the bulging mode yields solutions for $p < 3$, with $h/l$ decreasing as $p$ decreases. Thus, for all values of the ratio $h/l$, the bending mode is always more unstable than the bulging mode. In this respect, a rigid–plastic plate exhibits similar instability characteristics to an elastic plate. The results obtained here for a rigid–plastic plate under unaxial compression are quite different from earlier results based on bifurcation analysis for biaxial compression in plane strain. In the case of compression in plane strain, buckling occurs first in the bending mode as $h/l$ increases from zero, and then alternates back and forth from the bending mode to the bulging mode as $h/l$ continues to increase.

The eigencondition (5.9) for bending instability in a compressed Cosserat plate exhibits qualitatively similar behavior to a three-dimensional plate, inasmuch as $p$ increases as $h/l$ decreases. A value for $\zeta_3$ can be determined by matching the limit as $h/l \to \infty$ with the corresponding limit for a three-dimensional plate, that is, by putting

$$p = (3\zeta_3 - 1)/2 = 3,$$

which yields

$$\zeta_3 = 5/3. \tag{9.4}$$

For $h/l \to 0$, a comparison of (5.9) and (8.12) can be made by expanding (8.12) in a series in $(h^2/l^2)$. The right side of (8.12) has the limit of unity as $h/l \to 0$. The left side has the limit of unity as $p \to \infty$. Accordingly, the left side is expanded in powers of $(1/p)$. The leading terms of unity on each side cancel. The term in $(1/p)$ on the left includes the factor $(k_1^2 - k_2^2)$, which also appears in the term in $(h^2/l^2)$ on the right. When terms in $(1/p)^2$ and smaller are neglected on the left, and terms in $(h^2/l^2)^2$ and smaller neglected on the right, the result can be written

$$p = 36l^2/(\pi^2 h^2), \tag{9.5}$$

when it is noted that $(k_1^2 + 1)(k_2^2 + 1) = 1$ in the limit as $p \to \infty$. Thus, for the limiting case of a thin plate, the eigencondition (8.12) for a three-dimensional plate in the bending mode is independent of $H/P$. It has already been seen that the eigencondition (8.12) is independent of $H/P$ in the limit as $h/l \to \infty$. Thus, in the two limiting cases of a thick plate and a thin plate, (8.12) is independent of $H/P$, while (5.9) for a Cosserat plate is independent of $H/P$ for all $h/l$. The numerical factor of 36 on the right side of eqn (9.5) is 3 times the corresponding numerical factor on the right side of eqn (5.9). These two numerical factors can be brought into agreement by multiplying the constitutive coefficients $\zeta_5$, $\zeta_6$ and $\zeta_7$ by three, that is, by assigning the values

$$\zeta_5 = -16/h^2, \qquad \zeta_6 = \zeta_7 = 24/h^2, \tag{9.6}$$

to describe small bending deformations superposed on uniaxial compression.

For the bulging mode in a three-dimensional plate, there is a limiting minimum value of $h/l$ for which solutions exist. It has already been noted from the results of numerical computation that the eigencondition (8.13) has solutions in the range $0 < p < 3$, with $p$ decreasing as $h/l$ decreases. As $p \to 0$,

$$k_1^2 \sim 3(H/P - 1)/p, \qquad k_2^2 \sim -(H/P - 1)^{-1}. \tag{9.7}$$

The asymptotic behavior, as $p \to 0$, of the left side of eqn (8.13) is given by

$$\frac{1 - 3p^{-1}(k_2^2 + 1)^{-1}}{1 - 3p^{-1}(K_1^2 + 1)^{-1}} \sim -3/p, \tag{9.8}$$

while the behavior of the right side is given by

$$\frac{k_2}{k_1} \cdot \frac{\tanh[(k_2\pi h)/(2l)]}{\tanh[(k_1\pi h)/(2l)]} \sim (p/3)^{1/2}(H/P - 1)^{-1} \tan[(H/P - 1)^{-1/2}\pi h/(2l)]. \tag{9.9}$$

Thus the left side of eqn (8.13) approaches infinity as $p \to 0$. For the right side to approach infinity, the argument of the tangent function must approach $n\pi/2$, where $n$ is an odd integer. It follows that the limiting minimum value of $h/l$, for which a solution can be obtained as $p \to 0$, is given by

$$h/l = (H/P - 1)^{1/2}. \tag{9.10}$$

The eigencondition (6.12) for a Cosserat plate, compressed by load which satisifes (6.14), exhibits entirely different behavior. Solutions exist for all values of $h/l$, and the growth rate $p$ of the perturbation decreases as $h/l$ increases, while in a three-dimensional plate, $p$ decreases as $h/l$ decreases. Similar anomalous behavior, compared to a three-dimensional plate, was also noted in [8] for an elastic Cosserat plate. Also, there is no natural way of matching the eigenconditions for a Cosserat plate and a three-dimensional plate to determine the coefficient $\zeta_8$.

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